Chaos synchronization and parameter estimation from a scalar output signal

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(Received 29 May 2007; published 29 August 2007)

We propose an observer-based approach for chaos synchronization and parameter estimation from a scalar output signal. To begin with, we use geometric control to transform the master system into a standard form with zero dynamics. Then we construct a slaver to synchronize with the master using a combination of slide mode control and linear feedback control. Within a finite time, partial synchronization is realized, which further results in complete synchronization as time tends to infinity. Even if there exists model uncertainty in the slaver, we can also estimate the unknown model parameter by a simple adaptive rule.

DOI: 10.1103/PhysRevE.76.027203 PACS number(s): 82.40.Bj, 43.72.+q, 47.52.+j

Chaotic synchronization has been the focus of a growing literature during the last decade [1–10]. The original synchronization technique was developed by Pecora and Carroll [1]. In their seminal paper, they addressed the synchronization of chaotic systems in the drive-response coupling scheme. A chaotic system, called the master, generates a signal which is sent over a channel to a slaver, which uses this signal to synchronize itself with the master. Recently, chaos synchronization has been successfully used for the estimation of unknown model parameters in chaotic systems. The active-passive-decomposition method [11], the minimization strategy [12–14], the iterative method [15], and statistical methods [16] can be applied to estimate unknown model parameters in chaotic systems. Based on the well-known Lasalle invariance principle of differential equations, Huang and Guo also proposed an adaptive approach to estimate all unknown parameters [17]. Freitas *et al.* treated synchronization as a control problem and utilized geometric control to estimate unknown model parameters [18].

In this paper we also consider the estimation of unknown parameters in chaotic systems from a scalar output signal. This signal could be either a variable from the master or a scalar nonlinear combination of all variables from the master. We first transform the master into a standard form with zero dynamics by geometric control theory. Using the combination of slide mode control and linear feedback control, we then construct a slaver to synchronize itself with the master. Within a finite time, partial synchronization is first realized, which further results in complete synchronization as time tends to infinity. Even if there exists model uncertainty in the slaver, we can also estimate the unknown parameter by a simple adaptive rule. Presently, our strategy can estimate a single unknown parameter of the master, but this could be easily extended to many parameters.

Consider the chaotic systems

$$\dot{x} = f(x) + g(x)p,$$

$$s(t) = h(x), \tag{1}$$

where $f(x), g(x) \in R^n$ are nonlinear functions, $x \in R^n$ is the state, $s(t) \in R$ is the scalar output signal, and $p \in R$ is an unknown model parameter. Our problem is to identify the unknown parameter p from the time series $s(t)$. Throughout the paper, we assume that the parameter p is slowly varying: namely, $\dot{p} \approx 0$. Within the drive-response coupling scheme [1], system (1) is regarded as the master and the signal $s(t)$ is the driving signal to make a slaver synchronize with the master. Therefore, we should construct a suitable slaver, which can be used to estimate the unknown parameter p.

For completeness of this paper, we only introduce two concepts in the geometric control theory. A detailed background on geometric control is given in Refs. [18,19]. The first concept is the *Lie derivative operator* L—that is,

$$L_f^0 h(x) = h(x), \quad L_f^j h(x) = \sum_{i=1}^{n} \frac{\partial L_f^{j-1} h}{\partial x_i} f_i(x) \quad (j \ge 1). \tag{2}$$

The second concept is the *relative degree*. The scalar signal $h(x)$ has a relative degree r corresponding to the function $g(x)$ at point $x_0 \in M$, where M is a smooth manifold of dimension n, if

$$L_g L_f^{i-1} h(x) = 0, \tag{3}$$

for all x in a neighborhood of x_0 and all $i < r-1$, and

$$L_g L_f^{r-1} h(x_0) \neq 0. \tag{4}$$

If the signal $h(x)$ has a relative degree r for all $x \in M$, we can transform the master (1) into a standard form by a coordinate transformation [19]. This idea has been already used in Ref. [18]. However, there are several restrictions in this work. On the one hand, the estimation of the unknown parameter [namely, the controller $\tilde{u}$ given by Eq. (15) in Ref. [18]] requires all states of the master. On the other hand, the relative degree corresponding to the gain $H(x)$ of the control is n, which is denoted by Eq. (10) in Ref. [18]. In fact, within the driving-response scheme, it is better that only one scalar signal is required to drive the slaver, and the unknown parameter should be estimated by this scalar signal.

Let $z_1 = \phi_1(x) = h(x)$, $z_2 = \phi_2(x) = L_f h(x), \ldots, z_r = \phi_r(x) = L_f^{r-1} h(x)$. It can be shown [19] that there always exist $n-r$ functions ϕ_i $(i = r+1, \ldots, n)$ such that $L_g \phi_i(x) = 0$ and the Jacobian of the matrix function $\Phi(x) = [\phi_1(x), \ldots, \phi_n(x)]^T = [z_1, \ldots, z_n]^T$ is defined in the manifold M. Then the master (1) can be transformed into the following dynamics:

$$\dot{z}_j = z_{j+1}, \quad j = 1,2, \ldots, r-1,$$

$$\dot{z}_r = a(z,\eta) + b(z,\eta)p,$$

$$\dot{\eta} = f_0(z,\eta), \tag{5}$$

where the output

$$y = z_1,$$

$$z = (z_1, \ldots, z_r),$$

$$\eta = (z_{r+1}, \ldots, z_n),$$

$$a(z,\eta) = L_f^r h(\Phi^{-1}(z,\eta),\eta),$$

and

$$b(z,\eta) = L_g L_f^{r-1} h(\Phi^{-1}(z,\eta),\eta).$$

Moreover, the dynamics $\dot{\eta}=f_0(0,\eta)$ is called the *zero dynamics* [19], which means that $\lim_{t\to\infty}[\eta(t)-\eta'(t)]=0$ with $\eta(t)$ and $\eta'(t)$ being the solutions of $\dot{\eta}=f_0(\xi,\eta)$ and $\dot{\eta}'=f_0(\xi,\eta')$, respectively.

In fact, the meaning of *zero dynamics* has been already implied in a PC decomposition-based synchronization [1]. For an autonomous n-dimensional chaotic system $\dot{u}=f(u)$, we first divide the system into two subsystems $\dot{v}=g(v,w)$ and $\dot{w}=h(v,w)$, where $v=(u_1,\ldots,u_m)$, $g=(f_1(u),\ldots,f_m(u))$, $w=(u_{m+1},\ldots,u_n)$, and $h=(f_{m+1}(u),\ldots,f_n(u))$. Then we create a new subsystem $\dot{w}'=h(v,w')$, and we obtain an augmented system $\dot{v}=g(v,w)$, $\dot{w}=h(v,w), \dot{w}'=h(v,w')$. If the dynamics $\dot{w}=h(v,w)$ is asymptotically stable for all v, w' can approach w as time tends to infinity. It means that subsystems w and w' are synchronized with the help of the driving signal v. Generally speaking, if the driving signal is chosen suitably, many chaotic systems including the Lorenz system and Rössler system have zero dynamics (see Table I in Ref. [1]).

For the master (1) and its transformed form (5), we construct the model (a slaver)

$$\dot{\hat{z}}_j = \hat{z}_{j+1} + w_j \operatorname{sgn}(\xi_j - \hat{z}_j), \quad j = 1,2, \ldots, r-1,$$

$$\dot{\hat{z}}_r = a(\xi,\eta') + b(\xi,\eta')q + \Delta(\xi,\eta') + u,$$

$$\dot{\eta}' = f_0(\xi,\eta'), \tag{6}$$

with the estimated partial states

$$\xi_1 = s(t) = z_1,$$

$$\xi_{i+1} = \hat{z}_{i+1} + w_i \operatorname{sgn}(\xi_i - \hat{z}_i), \quad i = 1,2, \ldots, r-1, \tag{7}$$

where w_i $(i=1,2,\ldots,r-1)$ are parameters, $\Delta(\xi,\eta')$ is the model uncertainty, u is the external input, and the parameter q is adjusted by

$$\dot{q} = (\xi_r - \hat{z}_r)b(\xi,\eta'). \tag{8}$$

We assume that the model uncertainty $|\Delta(\xi,\eta')| \le \delta(\xi,\eta')d(t)$, where $\delta(\xi,\eta')$ is the known function and $d(t)$ is unknown but bounded time-varying disturbance. We construct a linear feedback control

$$u = k(\xi_r - \hat{z}_r) + \delta^2(\xi_r - \hat{z}_r), \tag{9}$$

where $k>0$.

Therefore we should choose suitable parameters w_i to make the partial synchronization be realized within a finite time; namely, the errors $e_i(t)=z_i(t)-\hat{z}_i(t)$ are zero after a finite time. Let $|e_i|_{\max}=\max_{t\ge 0}|e_i(t)|$. For $i=1,2,\ldots,r-1$, consider the sliding surface $\sigma_i: e_i=0$ and the Lyapunov function $V_i=\frac{1}{2}e_i^2$. It is clear that $\dot{V}_i=e_i e_{i+1}-w_i e_i \operatorname{sgn}(e_i)<0$ if w_i is chosen such that $w_i>|e_{i+1}|_{\max}$. Thus σ_i is attractive and can be reached in a finite time $T>0$. During the sliding surface σ_i, we also get $\dot{e}_i=0$. Hence it follows that for $i=1,2,\ldots,r-1$ and for all $t>T$,

$$0 = z_{i+1} - \hat{z}_{i+1} - w_i \operatorname{sgn}(e_i),$$

$$z_{i+1} = \hat{z}_{i+1} + w_i \operatorname{sgn}(e_i),$$

$$z_{i+1} = \xi_{i+1}. \tag{10}$$

A detailed proof of the attractiveness of the sliding surfaces σ_i and computation of the instant T are given in [6]. It is known that the time evolution of chaotic systems is confined to the chaotic attractor which can be estimated within a hyperellipsoid using the invariance theorem [20]. Let D_{i+1} be the orthogonal projection of the hyperellipsoid onto the $(i+1)$th axis of R^n space; then, D_{i+1} is a line segment of length L_{i+1}. It is clear that the maximum divergence of the error $e_{i+1}(t)$ is less than L_{i+1}. Therefore, $w_i=L_{i+1}$ is a suitable choice for the sliding observer gain. The idea of the choice of w_i is given in Refs. [6,20].

Equations (7) and (10) imply that partial synchronization in the systems (5) and (6) can be first realized within a finite time T. After the instant T, we obtain

$$\dot{e}_r = a(z,\eta) - a(\xi,\eta') + b(z,\eta)p - b(\xi,\eta')q - \Delta - ke_r - \delta^2 e_r,$$

$$\dot{e}_\eta = f_0(z,\eta) - f_0(\xi,\eta'), \tag{11}$$

where $e_\eta=\eta-\eta'$. From the above analysis, if we choose the suitable parameters w_i, we get $\xi_i=z_i$ after the instant T. Since the asymptotical stability of the zero dynamics $\dot{\eta}=f_0(\xi,\eta)$, the error e_η tends to zero as time approaches infinity. This means that $\dot{e}_r \approx b(\xi,\eta')p-b(\xi,\eta')q-\Delta-ke_r-\delta^2 e_r$ when time t is sufficiently large. For the error e_r, let a Lyapunov function $V_r=1/2e_r^2+1/2(p-q)^2$. When time t is sufficiently large, we get

$$\dot{V}_r \le -ke_r^2 - [\delta^2 e_r^2 - \delta d|e_r|] \le -ke_r^2 + d^2/4. \tag{12}$$

Therefore, the error e_r exponentially decays and is ultimately bounded. Since the parameters k can be freely adjusted, a transient performance and final estimation accuracy are guaranteed. Hence $q\approx p$ even if there exists a model uncertainty $\Delta(\xi,\eta')$ in the model.

From the above analysis, Eqs. (6)–(10) depend on the "sign" function. Fortunately, this function can be approximated by the continuous function

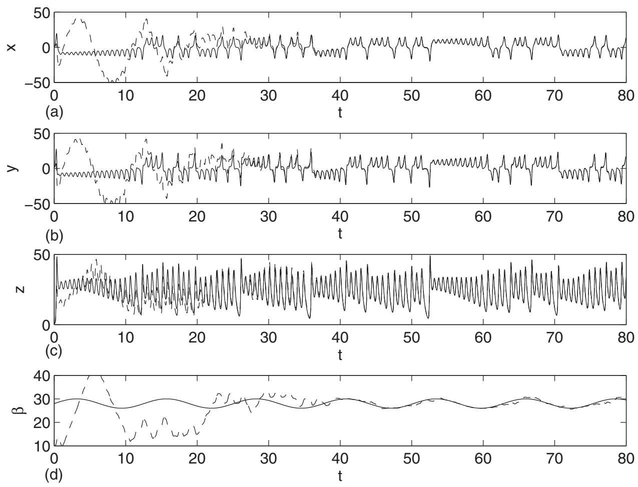

FIG. 1. Chaos synchronization and parameter estimation without uncertainty in the model (16). (a)–(c) show the curves of states of the master (solid lines) and the slaver (dashed lines). (d) shows the true parameter β (solid line) and the estimated parameter $\hat{\beta}$ (dashed line).

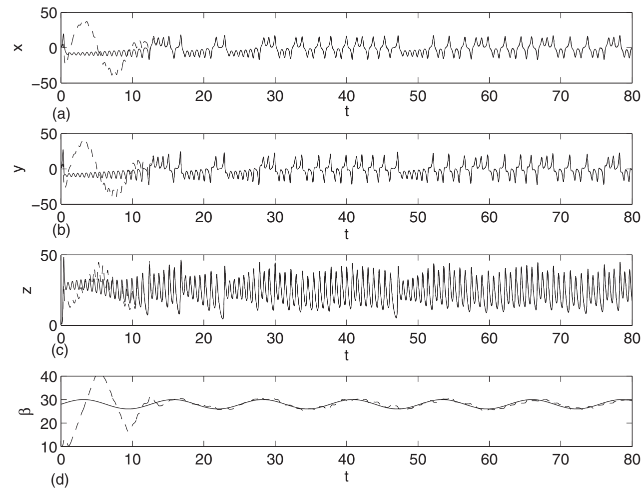

FIG. 2. Chaos synchronization and parameter estimation with uncertainty $\Delta(t)=|\eta|\sin(t)$ in the model (16). (a)–(c) show the curves of states of the master (solid lines) and the slaver (dashed lines). (d) shows the true parameter β (solid line) and the estimated parameter $\hat{\beta}$ (dashed line).

$$\mathrm{sgn}_\varepsilon(e_i) \approx e_i/(|e_i| + \varepsilon), \tag{13}$$

where $\varepsilon > 0$ is a sufficiently small number. For a suitable choice of ε, the continuous action (13) can approach the discontinuous sign function very well.

In order to verify our main results, we choose Lorenz system as an example. The dynamics of the Lorenz system is described by

$$\dot{x}_1 = \sigma(x_2 - x_1), \quad \dot{x}_2 = \beta x_1 - x_2 - x_1 x_3, \quad \dot{x}_3 = -\theta x_3 + x_1 x_2, \tag{14}$$

where σ, β, and θ are parameters. In order to show the efficiency of our approach, assume that the parameters σ and θ are known and the parameter β is unknown. Hence the Lorenz system becomes $\dot{x}=f(x)+g(x)\beta$ where $x=(x_1,x_2,x_3)$, $f(x)=(\sigma(x_2-x_1),-x_2-x_1x_3,-\theta x_3+x_1x_2)$, and $g(x)=(0,x_1,0)$. For the case of the driving signal $s(t)=h(x)=x_1$, we get $L_g h(x)=0$, $L_f h(x)=\sigma(x_2-x_1)$, and $L_g L_f h(x)=\sigma x_1$. Hence the scalar signal $h(x)=x_1$ has a relative degree $r=2$ corresponding to $g(x)$. In addition, we choose the transformation $z_1=\phi_1=x_1$, $z_2=\phi_2=\sigma(x_2-x_1)$, $z_3=\phi_3=x_3$. Then the transformed system is given by

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = a(z,\eta) + b(z,\eta)\beta, \quad \dot{\eta} = f_0(z,\eta) \tag{15}$$

where $z=(z_1,z_2)$, $\eta=z_3$, $a(z,\eta)=-\sigma z_1 z_3-\sigma z_1-(\sigma+1)z_2$, $b(z,\eta)=\sigma z_1$, and $f_0(z,\eta)=-\theta\eta+z_1^2+z_1z_2/\sigma$.

For the system (15), we construct a slaver

$$\dot{\hat{z}}_1 = \hat{z}_2 + w\,\mathrm{sgn}(\xi_1 - \hat{z}_1), \tag{16}$$

$$\dot{\hat{z}}_2 = a(\xi,\eta') + b(\xi,\eta')\hat{\beta} + \Delta(t) + u, \quad \dot{\eta}' = f_0(\xi,\eta'),$$

with the uncertainty $\Delta(t)$. Further, the estimated partial states $\xi_1=z_1$, $\xi_2=\hat{z}_2+w\,\mathrm{sgn}(\xi_1-\hat{z}_1)$ and the estimated parameter $\hat{\beta}$ is adjusted by $\dot{\hat{\beta}}=\sigma\xi_1(\xi_2-\hat{z}_2)$. In this simulation the parameters $\sigma=10$, $\theta=8/3$, and $\beta(t)=28+2\sin(0.5t)$. We first consider the case where there exists no uncertainty in the model (16)—i.e., $\Delta(t)=0$. In this case we choose a linear feedback control $u=k(\xi_2-\hat{z}_2)$ and the parameters $w=0.1$, $k=1.5$, and $\varepsilon=2$. Simulation results are shown in Fig. 1. The second case is that there is the uncertainty $\Delta(t)=|\eta|\sin(t)$ in the model (16). Hence $|\Delta(t)|=\delta d(t)$ with $\delta=|\eta|$ and $d(t)=1$. In this case we choose a linear feedback control $u=k(\xi_2-\hat{z}_2)+\delta^2(\xi_2-\hat{z}_2)$ and the parameters $w=0.1$, $k=1.5$, and $\varepsilon=2$. Simulation results are plotted in Fig. 2. From these figures, we show that our approach is very effective to estimate the unknown model parameter in chaotic systems.

Compared with Refs. [11–18], our approach has the following merits. (i) We can choose a complex nonlinear function $h(x)$ as a driving signal, provided that the transformation $\Phi(x)$ exists. The more complex this function is, the more secure the synchronization is. (ii) Even if the model (6) includes the model uncertainty, we can also choose a suitable parameter k to approximately estimate the unknown parameter. In the case of the uncertainty, the parameter estimation is difficult in Refs. [11,14,17,18]. (iii) The parameter estimation (8) is adjusted adaptively, and it only requires a scalar output signal. This may also estimate the time-varying parameter. (iv) The speed of partial synchronization can be adjusted by the parameters w_i $(i=1,2,\ldots,r)$. In real applications, we can choose suitable values to make the fast synchronization.

The authors would like to acknowledge the support of the Alexander von Humboldt Foundation.

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